

The asymptotics of an eigenfunction-correlation determinant for Dirac- δ perturbations

Martin Gebert

ABSTRACT. We prove the exact asymptotics of the scalar product of the ground states of two non-interacting Fermi gases confined to a 3-dimensional ball B_L of radius L in the thermodynamic limit, where the underlying one-particle operators differ by a Dirac- δ perturbation. More precisely, we show the algebraic decay of the correlation determinant $|\det(\langle\varphi_j^L, \psi_k^L\rangle)_{j,k=1,\dots,N}|^2 = L^{-\zeta(E)+o(1)}$, as $N, L \rightarrow \infty$ and $N/|B_L| \rightarrow \rho > 0$, where φ_j^L and ψ_k^L denote the lowest-energy eigenfunctions of the finite-volume one-particle Schrödinger operators. The decay exponent is given in terms of the s-wave scattering phase shift $\zeta(E) := \delta^2(\sqrt{E})/\pi^2$. For an attractive Dirac- δ perturbation we conclude that the decay exponent $\frac{1}{\pi^2} \|\arcsin |T(E)/2|\|_{\text{HS}}^2$ found in [GKMO14] does not provide a sharp upper bound on the decay of the correlation determinant.

1. Introduction

We consider the asymptotics of the scalar product of the ground states of two non-interacting finite-volume N -particle Schrödinger operators in the thermodynamic limit approaching the particle density $\rho(E) > 0$ corresponding to the Fermi energy $E > 0$. Here, the underlying one-particle Schrödinger operators are the negative Laplacian in 3-dimensional Euclidean space and the negative Laplacian with a Dirac- δ or zero-range perturbation sitting at the origin. We restrict this pair to the ball $B_L(0)$ of radius L and are interested in the L -asymptotics of the scalar product of the ground states of the corresponding two non-interacting N -particle operators, which we call ground-state overlap in the sequel. Using the representation of the ground states as Slater determinants, we see that the ground-state overlap is the following correlation determinant

$$\mathcal{S}_L^N := \det\left(\langle\varphi_j^L, \psi_k^L\rangle\right)_{1 \leq j, k \leq N}. \quad (1.1)$$

In this note, we are interested in its thermodynamic limit, i.e. increasing L and $N \in \mathbb{N}$ simultaneously such that $N/|B_L(0)| \rightarrow \rho(E) > 0$, where $\rho(E)$ denotes the integrated density of states of the negative Laplacian at the energy $E > 0$. Here, φ_j^L and ψ_k^L are the eigenfunctions belonging to the N lowest eigenvalues of the restricted operators, which we call H_L and $H_{\alpha,L}$, and $\langle \cdot, \cdot \rangle$ denotes the scalar

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product in $L^2(B_L(0))$. Anderson claimed in [And67] that in the case of a Dirac- δ -perturbation the determinant admits the asymptotics

$$|S_L^N|^2 \sim L^{-\zeta(E)} \quad (1.2)$$

as $N, L \rightarrow \infty$, $N/|B_L(0)| \rightarrow \rho(E) > 0$, where $\zeta(E) := \frac{1}{\pi^2} \delta^2(\sqrt{E})$ and δ refers to the s-wave scattering phase shift. This algebraic decay of the ground-state overlap is called Anderson's orthogonality catastrophe in the physics literature and we refer to [GKM14] for further references.

The starting point of the proofs of previous rigorous results is the following expansion of the determinant

$$\ln |S_L^N|^2 = - \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{tr} \left\{ \left(1_{(-\infty, \lambda_N^L]}(H_L) 1_{[\mu_{N+1}^L, \infty)}(H_{\alpha, L}) \right)^n \right\}, \quad (1.3)$$

valid for appropriate choices of N , where λ_N^L and μ_{N+1}^L denote the N th and $(N+1)$ th eigenvalue of the finite-volume operators H_L and $H_{\alpha, L}$, see [GKMO14]. Thus, estimates on the correlation determinant S_L^N are closely related to asymptotics of products of spectral projections given in (1.3). Considering only the $n=1$ term in (1.3), the first rigorous bounds on S_L^N were proved in [KOS13] valid for 1-dimensional systems and short-range perturbations. They found the upper bound on $|S_L^N|^2 \lesssim L^{-\tilde{\gamma}}$ with the decay exponent $\tilde{\gamma}(E) := \frac{1}{\pi^2} \|T(E)/2\|_{\text{HS}}^2$, where T refers to the scattering T -matrix of the corresponding infinite-volume operators, and a non-optimal lower bound. Later in [GKM14] the same upper bound $\tilde{\gamma}(E) := \frac{1}{\pi^2} \|T(E)/2\|_{\text{HS}}^2$ was deduced for quite general pairs of Schrödinger operators in arbitrary dimension, which differ by a sign-definite potential. Taking all summands in (1.3) into account, [GKMO14] proved an upper bound with the decay exponent $\gamma(E) := \frac{1}{\pi^2} \|\arcsin |T(E)/2|\|_{\text{HS}}^2$ in the general setting discussed in [GKM14]. Let us point out that these previous results concern upper bounds and are also valid for special choices of thermodynamic limits only. Here, in the toy model of a Dirac- δ perturbation we provide the exact asymptotics of the correlation determinant and we consider arbitrary thermodynamic limits approaching a particle density $\rho > 0$, see Theorem 2.1 below. We show this using a representation of the ground-state overlap other than (1.3), which is valid for rank-1-perturbations, i.e.

$$|S_L^N|^2 = \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{|\mu_k^L - \lambda_j^L| |\lambda_k^L - \mu_j^L|}{|\lambda_k^L - \lambda_j^L| |\mu_k^L - \mu_j^L|}, \quad (1.4)$$

where λ_k^L and μ_j^L are the eigenvalues of the pair of the finite-volume Schrödinger operators, see Section 3. This formula is known in physics literature and goes back at least to [TO85]. Using the latter formula, we give a straightforward proof of the algebraic decay (1.2) with the exponent $\zeta(E) = \frac{1}{\pi^2} \delta^2(\sqrt{E})$, as Anderson predicted. It turns out that the decay exponent is equal to the one found in [GKMO14] in the case of a repulsive Dirac- δ perturbation only, i.e. $\zeta(E) = \frac{1}{\pi^2} \|\arcsin |T(E)/2|\|_{\text{HS}}^2$. On the other hand, we obtain $\zeta(E) > \frac{1}{\pi^2} \|\arcsin |T(E)/2|\|_{\text{HS}}^2$, for an attractive Dirac- δ see Remark 2.3 below. Hence, the decay exponent $\frac{1}{\pi^2} \|\arcsin |T(E)/2|\|_{\text{HS}}^2$ does not provide the exact asymptotics of (1.1).

Recently, [KOS15] proved the asymptotics of a shifted correlation determinant for one-dimensional models with a perturbation by a magnetic field. A related

problem, which we mention for completeness, is considering the asymptotics of products of spectral projections of infinite-volume operators, similar to (1.3). This was done in the proof of [GKMO14] and extended in [FP14].

2. Model and results

We start with the operator $-\Delta_0 : C_c^\infty(\mathbb{R}^3 \setminus \{0\}) \rightarrow L^2(\mathbb{R}^3)$, which has deficiency indices $(1, 1)$. Therefore, $-\Delta_0$ gives rise to a one-parameter family of self-adjoint extensions which we index by $\alpha \in \mathbb{R}$ and denote by $-\Delta_\alpha$, see [AGHH05, Chapter 1]. We refer to $-\Delta_\alpha$ as the negative Laplacian with a Dirac- δ perturbation sitting at the origin 0 of strength α . Throughout, we consider for $\alpha \in \mathbb{R}$ the pair of Schrödinger operators

$$H := -\Delta \quad \text{and} \quad H_\alpha := -\Delta_\alpha \quad (2.1)$$

on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^3)$, where $-\Delta$ is the negative Laplacian. More precisely, following [AGHH05, Chapter 1], the operators H and H_α admit a decomposition with respect to angular momentum. Thus, there exists a unitary U such that both operators transform into the direct sum

$$UHU^* = \bigoplus_{\ell \in \mathbb{N}_0} \bigoplus_{-\ell \leq m \leq \ell} h^\ell \quad \text{and} \quad UH_\alpha U^* = \bigoplus_{\ell \in \mathbb{N}_0} \bigoplus_{-\ell \leq m \leq \ell} h_\alpha^\ell, \quad (2.2)$$

where $h_{(\alpha)}^\ell : L^2((0, \infty)) \supset \text{dom}(h_{(\alpha)}^\ell) \rightarrow L^2((0, \infty))$ and $h^\ell = h_\alpha^\ell$ for all $\ell \geq 1$. In the $\ell = 0$ case the operators are given by

$$h^0 = -\frac{d^2}{dx^2}, \quad \text{dom}(h^0) = \{f \in L^2((0, \infty)) : f, f' \in AC_{\text{loc}}((0, \infty)); \quad (2.3)$$

$$f(0+) = 0; f'' \in L^2((0, \infty))\}$$

$$h_\alpha^0 = -\frac{d^2}{dx^2}, \quad \text{dom}(h_\alpha^0) = \{f \in L^2((0, \infty)) : f, f' \in AC_{\text{loc}}((0, \infty)); \quad (2.4)$$

$$-4\pi\alpha f(0+) + f'(0+) = 0; f'' \in L^2((0, \infty))\},$$

where we denote by $AC_{\text{loc}}((0, \infty))$ the set of all locally absolutely continuous functions. Thus, the difference of H and H_α takes place in the lowest angular momentum channel via a different boundary condition at 0 which we parametrise by $\alpha \in \mathbb{R}$. In the following we are interested in the restrictions of these operators to the ball $B_L(0)$ of radius L around the origin

$$H_L := -\Delta_L \quad \text{and} \quad H_{\alpha,L} := -\Delta_{\alpha,L}. \quad (2.5)$$

Here, $-\Delta_L$ is the negative Dirichlet Laplacian on $B_L(0)$. The operator $-\Delta_{\alpha,L}$ corresponds to the restriction of the operator $-\Delta_\alpha$ imposing Dirichlet boundary condition at L in each angular momentum channel, i.e. also the restriction of $-\Delta_\alpha$ to $B_L(0)$ with Dirichlet boundary conditions. Thus, H_L and $H_{\alpha,L}$ differ as well as before in the lowest angular momentum channel only by a different boundary condition at 0. We call the corresponding operators in the $\ell = 0$ channel, i.e the restrictions of h^0 and h_α^0 to the interval $(0, L)$ with Dirichlet boundary condition at L ,

$$h_L^0 \quad \text{and} \quad h_{\alpha,L}^0. \quad (2.6)$$

Using standard results for regular Sturm-Liouville operators, we obtain for all $z \in \rho(h_L^0) \cap \rho(h_{\alpha,L}^0)$ a vector $\eta_{L,z}^\alpha \in L^2(B_L(0))$ such that the resolvents satisfy

$$\frac{1}{h_L^0 - z} - \frac{1}{h_{\alpha,L}^0 - z} = |\eta_{L,z}^\alpha\rangle\langle\eta_{L,z}^\alpha|. \quad (2.7)$$

Thus, $h_{\alpha,L}^0$ is a rank-1-perturbation of h_L^0 in the resolvent, and the same is true for the pair $H_{\alpha,L}$ and H_L . We point out that the perturbation is not compactly supported since $\eta_{L,z}^\alpha$ is L dependent. Moreover, the compactness of the resolvents of H_L and $H_{\alpha,L}$ imply that both H_L and $H_{\alpha,L}$ have discrete spectra. We write

$$\lambda_1^L \leq \lambda_2^L \leq \dots \quad \text{and} \quad \mu_1^L \leq \mu_2^L \leq \dots \quad (2.8)$$

for their non-decreasing sequences of eigenvalues, counting multiplicities, and $(\varphi_j^L)_{j \in \mathbb{N}}$ and $(\psi_k^L)_{k \in \mathbb{N}}$ for the corresponding sequences of normalised eigenfunctions, where we choose the same eigenvectors for H_L and $H_{\alpha,L}$ in any angular momentum channel $\ell \geq 1$. This choice ensures that the eigenfunctions of H_L and $H_{\alpha,L}$ differ in the lowest angular momentum channel only. Let us point out that in the case of $\alpha < 0$ there exists precisely one negative eigenvalue $\mu_1 = -(4\pi\alpha)^2$ for the infinite-volume operator H_α , respectively h_α^0 , see [AGHH05, Chapter 1]. Dirichlet-Neumann bracketing implies $h_\alpha^0 \leq h_{\alpha,L}^0 \oplus h_{L^c}^0$, where $h_{L^c}^0$ denotes the negative Laplacian on (L, ∞) with Dirichlet boundary condition at L . Thus, in the case of $\alpha < 0$ we obtain the uniform lower bound on the finite-volume operators

$$H_{\alpha,L} \geq -(4\pi\alpha)^2 \quad \text{and equivalently} \quad h_{\alpha,L}^0 \geq -(4\pi\alpha)^2. \quad (2.9)$$

Let $N \in \mathbb{N}$. In the following we are interested in the correlation determinant

$$\mathcal{S}_L^N := \det\left(\langle \varphi_j^L, \psi_k^L \rangle\right)_{1 \leq j, k \leq N}. \quad (2.10)$$

The main result concerning \mathcal{S}_L^N is the following.

Theorem 2.1. *Let $\alpha \in \mathbb{R}$, $E > 0$ and $N_{(\cdot)}(E) : \mathbb{R}_+ \rightarrow \mathbb{N}$ an arbitrary function subject to*

$$\frac{N_L(E)}{|B_L(0)|} \rightarrow \rho(E) := \frac{E^{3/2}}{8\pi^3}, \quad (2.11)$$

i.e. ρ denotes the integrated density of states of the operator $-\Delta$. Then, the correlation determinant corresponding to the pair H_L and $H_{\alpha,L}$ admits the asymptotics

$$|\mathcal{S}_L^{N_L(E)}|^2 = L^{-\frac{1}{\pi^2}\delta_\alpha^2(\sqrt{E})+o(1)}, \quad (2.12)$$

as $L \rightarrow \infty$, equivalently,

$$\lim_{L \rightarrow \infty} \frac{\ln |\mathcal{S}_L^{N_L(E)}|}{\ln L} = -\frac{1}{2\pi^2}\delta_\alpha^2(\sqrt{E}), \quad (2.13)$$

and δ_α is given by Definition 2.2 below.

Definition 2.2 (Scattering phase shift). Let $k > 0$. Then, the scattering phase shift is defined by

$$\delta_\alpha(k) := \begin{cases} \arctan\left(\frac{k}{4\pi\alpha}\right) & \text{for } \alpha \geq 0, \\ \pi - \arctan\left(\frac{k}{4\pi|\alpha|}\right) & \text{for } \alpha \leq 0, \end{cases} \quad (2.14)$$

where we use the convention $\arctan\left(\frac{k}{0}\right) := \frac{\pi}{2}$ for $k > 0$.

Remarks 2.3. (i) The separate definitions of the phase shift are reminiscent to the existence of a negative eigenvalue whenever $\alpha < 0$ and Levinson's theorem.

(ii) Due to the nature of a Dirac- δ perturbation in 3 dimensions the same result is apparently valid for the corresponding problem on the half-axis.

(iii) We emphasise that we allow arbitrary thermodynamic limits approaching the particle density $\rho > 0$.

(iv) The $o(1)$ -error in (2.12) depends on the particular choice of the thermodynamic limit. To see this, we refer to equations (4.38) and (4.39) in the proof of Theorem 2.1. In particular, we think that the error cannot be improved allowing arbitrary thermodynamic limits.

(v) In [GKMO14, Theorem 2.2] an upper bound on the ground-state overlap is proved for quite general pairs of Schrödinger operators which is valid for subsequences only. More precisely, they prove for a subsequence

$$\limsup_{L \rightarrow \infty} \frac{\ln |\mathcal{S}_L^{N_L(E)}|}{\ln L} \leq -\frac{\gamma(E)}{2}, \quad (2.15)$$

where

$$\gamma(E) := \frac{1}{\pi^2} \|\arcsin |T(E)/2|\|_{\text{HS}}^2 \quad (2.16)$$

and T denotes the scattering T -matrix. Since we consider here s-wave scattering, we restrict ourselves to the lowest angular momentum channel. In this case, $T(E)$ is a complex number and $|T(E)/2| = \sin(\delta_\alpha(\sqrt{E}))$. Now, computing $\gamma(E)$ yields

$$\gamma(E) = \begin{cases} \frac{1}{\pi^2} \delta_\alpha^2(\sqrt{E}) & \text{for } |\delta_\alpha(\sqrt{E})| \leq \frac{\pi}{2} \\ \frac{1}{\pi^2} (\arcsin(\sin(\delta_\alpha(\sqrt{E})))^2 & \text{for } |\delta_\alpha(\sqrt{E})| \geq \frac{\pi}{2}. \end{cases} \quad (2.17)$$

Thus, in general the decay exponent $\gamma(E)$ does not provide a sharp upper bound on the correlation determinant whenever the phase shift is bigger than $\pi/2$. In our model this is equivalent to $\alpha < 0$ which we refer to as the attractive case.

The proof of Theorem 2.1 follows from a different approach than the one made in [GKM14] and [GKMO14], i.e. we do not use the representation (1.3) in this article. Here, the key is the following remarkable product representation of the determinant in terms of the eigenvalues of the finite-volume Schrödinger operators. To our knowledge, this was first stated in [TO85].

Lemma 2.4. *Let $N \in \mathbb{N}$. Then,*

$$\left| \det \left(\langle \varphi_j^L, \psi_k^L \rangle \right)_{1 \leq j, k \leq N} \right|^2 = \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{|\mu_k^L - \lambda_j^L| |\lambda_k^L - \mu_j^L|}{|\lambda_k^L - \lambda_j^L| |\mu_k^L - \mu_j^L|}. \quad (2.18)$$

We start with proving this product representation for general pairs of compact operators which differ by a rank-1-perturbation in Section 3. We apply this to our setting in Section 4 and prove Theorem 2.1.

3. Representation of the ground-state overlap

In this section we prove a quite general representation for determinants of eigenvectors of pairs of operators which differ by a rank-1-perturbation. The main result in this section, Theorem 3.1, will be the key to the proof of Theorem 2.1.

Let \mathcal{H} be a separable infinite-dimensional Hilbert space and $A : \mathcal{H} \rightarrow \mathcal{H}$ be a compact, linear and self-adjoint operator. Moreover, we assume $A \geq 0$ with $\ker(A) = \{0\}$. We define

$$B := A + |\phi\rangle\langle\phi| \quad (3.1)$$

for some $0 \neq \phi \in \mathcal{H}$. We write $\alpha_1 \geq \alpha_2 \geq \dots$ and $\beta_1 \geq \beta_2 \geq \dots$ for the non-increasing sequences of eigenvalues of A , respectively B and denote by $(\varphi_j)_{j \in \mathbb{N}}$ and $(\psi_k)_{k \in \mathbb{N}}$ the corresponding normalised eigenvectors. Since A and B differ by a rank-1-perturbation, the min-max theorem implies that the eigenvalues interlace. We assume the following condition on the eigenvalues

$$\sum_{n=1}^{\infty} |\beta_n - \alpha_n| < \infty. \quad (3.2)$$

Moreover, for simplicity we also assume the following strict interlacing condition

$$\beta_1 > \alpha_1 > \beta_2 > \alpha_2 > \dots \quad (3.3)$$

In particular, $\beta_k \neq \alpha_j$ for all $j, k \in \mathbb{N}$. Furthermore, the above implies cyclicity of ϕ . Assumption (3.3) is not necessary but simplifies notation and computations. In the general case one has to consider the restriction to the cyclic subspace generated by the perturbation ϕ . But the application in mind will satisfy the interlacing condition (3.3), therefore, we assume it.

Theorem 3.1. *Let $N \in \mathbb{N}$. We assume conditions (3.2) and (3.3) to hold. Then,*

$$\left| \det \left(\langle \varphi_j, \psi_k \rangle \right)_{1 \leq j, k \leq N} \right|^2 = \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{|\beta_k - \alpha_j| |\alpha_k - \beta_j|}{|\alpha_k - \alpha_j| |\beta_k - \beta_j|}. \quad (3.4)$$

PROOF OF THEOREM 3.1. We use the eigenvalue equations and assumption (3.3) to obtain for all $j, k \in \mathbb{N}$

$$\langle \varphi_j, \psi_k \rangle = \frac{\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle}{\beta_k - \alpha_j}. \quad (3.5)$$

Hence, the multi-linearity of the determinant implies

$$\begin{aligned} & \left| \det \left(\langle \varphi_j, \psi_k \rangle \right)_{1 \leq j, k \leq N} \right|^2 \\ &= \left| \det \left(\frac{\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle}{\beta_k - \alpha_j} \right)_{1 \leq j, k \leq N} \right|^2 \\ &= \left(\prod_{j=1}^N \prod_{k=1}^N |\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle|^2 \right) \left| \det \left(\frac{1}{\beta_k - \alpha_j} \right)_{1 \leq j, k \leq N} \right|^2. \end{aligned} \quad (3.6)$$

Now, the remaining determinant can be computed explicitly. We use the Cauchy determinant formula to evaluate this, see e.g. [Wey13, Lem. 7.6.A], and end up with

$$(3.6) = \left(\prod_{j=1}^N \prod_{k=1}^N |\langle \varphi_j, \phi \rangle \langle \phi, \psi_k \rangle|^2 \right) \frac{\prod_{j,k=1, j \neq k}^N |\beta_k - \beta_j| |\alpha_j - \alpha_k|}{\prod_{j,k=1}^N |\beta_k - \alpha_j|^2}. \quad (3.7)$$

Corollary 3.3 below yields

$$\begin{aligned} (3.7) &= \left(\prod_{k=1}^N \prod_{\substack{l=1 \\ l \neq k}}^{\infty} \frac{|\alpha_l - \beta_k|}{|\beta_l - \beta_k|} \right) \left(\prod_{j=1}^N \prod_{\substack{l=1 \\ l \neq j}}^{\infty} \frac{|\beta_l - \alpha_j|}{|\alpha_l - \alpha_j|} \right) \prod_{\substack{j,k=1 \\ j \neq k}}^N \frac{|\beta_k - \beta_j| |\alpha_j - \alpha_k|}{|\beta_k - \alpha_j|^2} \\ &= \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{|\beta_k - \alpha_j| |\alpha_k - \beta_j|}{|\beta_k - \beta_j| |\alpha_j - \alpha_k|}. \end{aligned} \quad (3.8)$$

This gives the claim, where we remark that by assumption (3.2) all products in the latter converge absolutely. \square

To complete the proof, we continue with computing the resolvents of the operators A and B in terms of their eigenvalues.

Lemma 3.2. *We assume (3.2) and (3.3). Then, there exist $a, b \in \mathbb{R}$ with $ab = -1$ such that*

(i) *for all $z \in \varrho(A)$*

$$\left\langle \phi, \frac{1}{A - z} \phi \right\rangle + 1 = a \prod_{k=1}^{\infty} \frac{\beta_k - z}{\alpha_k - z}, \quad (3.9)$$

(ii) *for all $z \in \varrho(B)$*

$$\left\langle \phi, \frac{1}{B - z} \phi \right\rangle - 1 = b \prod_{n=1}^{\infty} \frac{\alpha_n - z}{\beta_n - z}. \quad (3.10)$$

Corollary 3.3. *Let $j, k \in \mathbb{N}$. Under the assumption (3.2) and (3.3)*

$$|\langle \varphi_j, \phi \rangle \langle \psi_k, \phi \rangle|^2 = |\beta_j - \alpha_j| |\alpha_k - \beta_k| \left(\prod_{\substack{l=1 \\ l \neq j}}^{\infty} \frac{|\beta_l - \alpha_j|}{|\alpha_l - \alpha_j|} \right) \left(\prod_{\substack{l=1 \\ l \neq k}}^{\infty} \frac{|\alpha_l - \beta_k|}{|\beta_l - \beta_k|} \right). \quad (3.11)$$

PROOF OF COROLLARY 3.3. Using Lemma 3.2 we compute the residue of the resolvents

$$\begin{aligned} |\langle \varphi_j, \phi \rangle|^2 &= \lim_{z \rightarrow \alpha_j} (\alpha_j - z) \left\langle \phi, \frac{1}{A - z} \phi \right\rangle \\ &= \lim_{z \rightarrow \alpha_j} (\alpha_j - z) a \prod_{l=1}^{\infty} \frac{(\beta_l - z)}{(\alpha_l - z)} = a (\beta_j - \alpha_j) \prod_{\substack{l=1 \\ l \neq j}}^{\infty} \frac{(\beta_l - \alpha_j)}{(\alpha_l - \alpha_j)} \end{aligned} \quad (3.12)$$

and along the same line

$$|\langle \psi_k, \phi \rangle|^2 = b (\alpha_k - \beta_k) \prod_{\substack{l=1 \\ l \neq k}}^{\infty} \frac{(\alpha_l - \beta_k)}{(\beta_l - \beta_k)}. \quad (3.13)$$

Taking the absolute value and using $|ab| = 1$, we get the result. \square

PROOF OF LEMMA 3.2. First note that by assumption (3.2) the sequences

$$\left(\prod_{k=1}^N \frac{\beta_k - z}{\alpha_k - z} \right)_{N \in \mathbb{N}} \quad \text{and} \quad \left(\prod_{n=1}^N \frac{\alpha_n - z}{\beta_n - z} \right)_{N \in \mathbb{N}} \quad (3.14)$$

converge locally uniformly for all $z \in \varrho(A) \cap \varrho(B)$, see [Kno96, Thm. 252]. Therefore, the limits

$$F(z) := \prod_{n=1}^{\infty} \frac{\alpha_n - z}{\beta_n - z} \quad \text{and} \quad G(z) := \prod_{k=1}^{\infty} \frac{\beta_k - z}{\alpha_k - z} \quad (3.15)$$

are well-defined analytic functions on $\varrho(A) \cap \varrho(B)$, which fulfil $FG = 1$. Due to the locally uniform convergence, the derivative of F satisfies

$$\begin{aligned} F'(z) &= \lim_{N \rightarrow \infty} \sum_{l=1}^N \prod_{\substack{n=1 \\ n \neq l}}^N \frac{\alpha_n - z}{\beta_n - z} \frac{d}{dz} \frac{\alpha_l - z}{\beta_l - z} \\ &= \lim_{N \rightarrow \infty} \sum_{l=1}^N \prod_{\substack{n=1 \\ n \neq l}}^N \frac{\alpha_n - z}{\beta_n - z} \frac{\alpha_l - \beta_l}{(\beta_l - z)^2} = F(z) \lim_{N \rightarrow \infty} \sum_{l=1}^N \left(\frac{1}{\beta_l - z} - \frac{1}{\alpha_l - z} \right) \end{aligned} \quad (3.16)$$

for all $z \in \varrho(A) \cap \varrho(B)$. We apply Lemma 3.4 below and obtain

$$(3.16) = -F(z) \left\langle \frac{1}{A-z} \phi, \frac{1}{B-z} \phi \right\rangle. \quad (3.17)$$

Now, the resolvent identity implies for all $z \in \varrho(A) \cap \varrho(B)$

$$\frac{1}{B-z} - \frac{1}{A-z} = -\frac{1}{A-z} \phi \left\langle \frac{1}{B-\bar{z}} \phi, \cdot \right\rangle \quad (3.18)$$

which provides the equality

$$\frac{1}{A-z} \phi = \frac{1}{1 - \langle \frac{1}{B-\bar{z}} \phi, \phi \rangle} \frac{1}{B-z} \phi. \quad (3.19)$$

Inserting this into (3.17), we see that F solves the differential equation

$$F'(E) = F(E) \frac{1}{\langle \phi, \frac{1}{B-E} \phi \rangle - 1} \left\langle \phi, \left(\frac{1}{B-E} \right)^2 \phi \right\rangle \quad (3.20)$$

at least for all $E \in \varrho(A) \cap \varrho(B) \cap \mathbb{R}$. On the other hand the resolvent of B is analytic in $\varrho(B)$ and the function $t \mapsto \langle \phi, \frac{1}{B-t} \phi \rangle - 1$, $t < 0$, solves the above ODE (3.20) as well. Now, the general solution to this ODE is $f(t) = x_0 \exp \left(\int_{t_0}^t ds \frac{1}{\langle \phi, \frac{1}{B-s} \phi \rangle - 1} \left\langle \phi, \left(\frac{1}{B-s} \right)^2 \phi \right\rangle \right)$, for some initial condition (t_0, x_0) . Note that the functions $t \mapsto F(t)$ and $t \mapsto \langle \phi, \frac{1}{B-t} \phi \rangle - 1$ are non-zero, thus $\langle \phi, \frac{1}{B-t} \phi \rangle - 1 = cF(t)$ for some $c \neq 0$. This and the identity theorem for analytic functions give the claim. Equation (3.9) follows from $F(z)G(z) = 1$ and the identity

$$\left(\left\langle \phi, \frac{1}{B-z} \phi \right\rangle - 1 \right) \left(\left\langle \phi, \frac{1}{A-z} \phi \right\rangle + 1 \right) = -1, \quad (3.21)$$

for all $z \in \varrho(A) \cap \varrho(B)$ which is a consequence of (3.18). \square

Lemma 3.4. *Let $z \in \varrho(A) \cap \varrho(B)$. Assume (3.3). Then, we obtain the following identity*

$$\lim_{N \rightarrow \infty} \sum_{l=1}^N \left(\frac{1}{\beta_l - z} - \frac{1}{\alpha_l - z} \right) = - \left\langle \frac{1}{A - z} \phi, \frac{1}{B - z} \phi \right\rangle. \quad (3.22)$$

Let us point out that in the finite-dimensional case the above equality follows directly from the resolvent equation, (3.18). Nevertheless, the infinite-dimensional case is a bit more involved due to convergence issues.

PROOF. For $\lambda \in \mathbb{R}$ we define the operator

$$A(\lambda) := A + \lambda |\phi\rangle\langle\phi| \quad (3.23)$$

and write $\alpha_l(\lambda)$ for the l th eigenvalue counted from above and $\varphi_l(\lambda)$ for the corresponding eigenvector. Moreover, we remark that $\alpha_l(1)$ and $\varphi_l(1)$ correspond to β_l and ψ_l . Assumption (3.3) and the definite sign of the perturbation imply that the eigenvalues of $A(\lambda)$ are non-degenerate for all $\lambda \in [0, 1]$. Thus, standard results, see [RS78, Chap. XII], give differentiability of the eigenvalues for all $\lambda \in (0, 1)$ and we apply the Feynman-Hellmann theorem, see e.g. [IZ88], to deduce for all $l \in \mathbb{N}$ and $\lambda \in (0, 1)$

$$\alpha'_l(\lambda) = |\langle \varphi_l(\lambda), \phi \rangle|^2. \quad (3.24)$$

Hence, we compute using the latter

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{l=1}^N \left(\frac{1}{\beta_l - z} - \frac{1}{\alpha_l - z} \right) &= - \lim_{N \rightarrow \infty} \sum_{l=1}^N \int_0^1 d\lambda \left(\frac{1}{\alpha_l(\lambda) - z} \right)^2 \alpha'_l(\lambda) \\ &= - \lim_{N \rightarrow \infty} \sum_{l=1}^N \int_0^1 d\lambda \left(\frac{1}{\alpha_l(\lambda) - z} \right)^2 |\langle \varphi_l(\lambda), \phi \rangle|^2. \end{aligned} \quad (3.25)$$

The eigenvalue equation implies

$$\begin{aligned} (3.25) &= - \lim_{N \rightarrow \infty} \sum_{l=1}^N \int_0^1 d\lambda \left\langle \frac{1}{A(\lambda) - \bar{z}} \phi, \varphi_l(\lambda) \right\rangle \left\langle \varphi_l(\lambda), \frac{1}{A(\lambda) - z} \phi \right\rangle \\ &= - \int_0^1 d\lambda \left\langle \phi, \left(\frac{1}{A(\lambda) - z} \right)^2 \phi \right\rangle, \end{aligned} \quad (3.26)$$

where we used Fubini's theorem to interchange the integral with the sum and the fact that the vectors $(\varphi_l(\lambda))_{l \in \mathbb{N}}$ form an ONB. The resolvent identity (3.18) implies

$$\frac{1}{A(\lambda) - z} \phi = \frac{1}{1 + \lambda \langle \phi, \frac{1}{A - z} \phi \rangle} \frac{1}{A - z} \phi. \quad (3.27)$$

Therefore, we continue

$$\begin{aligned}
(3.26) &= - \int_0^1 d\lambda \left\langle \phi, \left(\frac{1}{A-z} \right)^2 \phi \right\rangle \left(\frac{1}{1 + \lambda \langle \phi, \frac{1}{A-z} \phi \rangle} \right)^2 \\
&= - \frac{\langle \phi, \left(\frac{1}{A-z} \right)^2 \phi \rangle}{\langle \phi, \frac{1}{A-z} \phi \rangle} \int_0^1 d\lambda \frac{d}{d\lambda} \left(\frac{1}{1 + \lambda \langle \phi, \frac{1}{A-z} \phi \rangle} \right) \\
&= \frac{\langle \phi, \left(\frac{1}{A-z} \right)^2 \phi \rangle}{\langle \phi, \frac{1}{A-z} \phi \rangle} \left(1 - \left(\frac{1}{1 + \langle \phi, \frac{1}{A-z} \phi \rangle} \right) \right) = - \frac{\langle \phi, \left(\frac{1}{A-z} \right)^2 \phi \rangle}{1 + \langle \phi, \frac{1}{A-z} \phi \rangle}. \quad (3.28)
\end{aligned}$$

Equation (3.27) with $\lambda = 1$ provides the assertion

$$(3.28) = - \left\langle \frac{1}{A-z}, \frac{1}{B-z} \phi \right\rangle. \quad (3.29)$$

□

4. Proof of Theorem 2.1

We decompose the determinant according to the angular momentum decomposition (2.2). This implies

$$\left| \det \left(\langle \varphi_j^L, \psi_k^L \rangle \right)_{1 \leq j, k \leq N_L(E)} \right|^2 = \prod_{l \in \mathbb{N}_0} \left| \det \left(\langle \varphi_j^L(\ell), \psi_k^L(\ell) \rangle \right)_{1 \leq j, k \leq N_L^l(E)} \right|^{2(2\ell+1)}, \quad (4.1)$$

where $\varphi_j^L(\ell)$ and $\psi_k^L(\ell)$ correspond to the radial part of the eigenfunctions lying in the ℓ -th angular momentum channel and $N_L^l(E)$ to the relative particle number in the ℓ -th angular momentum channel. More precisely,

$$N_L^l(E) := \# \{ k \in \mathbb{N} : \exists j \in \{1, \dots, N_L\} \text{ with } \lambda_k^L(\ell) = \lambda_j^L \} \quad (4.2)$$

where $(\lambda_k^L(\ell))_{k \in \mathbb{N}}$ denote the eigenvalues of h_L^ℓ . Since we chose the eigenfunctions of H_L and $H_{\alpha, L}$ to be the same in every angular momentum channel $\ell \geq 1$ we obtain that only the $\ell = 0$ term in the product (4.1) is different from 1. Hence,

$$\left| \det \left(\langle \varphi_j^L, \psi_k^L \rangle \right)_{1 \leq j, k \leq N_L(E)} \right|^2 = \left| \det \left(\langle \varphi_j^L(0), \psi_k^L(0) \rangle \right)_{1 \leq j, k \leq N_L^0(E)} \right|^2, \quad (4.3)$$

Thus, we reduced our problem to a problem on the half-axis, where the relative particle number satisfies

Lemma 4.1. *Given $E > 0$. Let L and $N_L(E) \in \mathbb{N}$ such that $\frac{N_L(E)}{|B_L(0)|} \rightarrow \rho(E)$ as $L \rightarrow \infty$. Then,*

$$\frac{N_L^0(E)}{L} \rightarrow \frac{\sqrt{E}}{\pi} =: \rho_0(E), \quad (4.4)$$

as $L \rightarrow \infty$.

PROOF. For any $E > 0$

$$\lim_{L \rightarrow \infty} \frac{\#\{k : \lambda_k^L \leq E\}}{|B_L(0)|} = \rho(E) = \lim_{L \rightarrow \infty} \frac{N_L(E)}{|B_L(0)|}, \quad (4.5)$$

where the first equality follows from e.g. [RS78]. Hence, we obtain for an arbitrary $\epsilon > 0$ the inequalities

$$\#\{k : \lambda_k^L \leq E - \epsilon\} \leq N_L(E) \leq \#\{k : \lambda_k^L \leq E + \epsilon\} \quad (4.6)$$

for L large enough. Since ρ is strictly increasing, we obtain $\lambda_{N_L(E)}^L \rightarrow E$. Therefore, $\lambda_{N_L^0(E)}^L(0) \rightarrow E$ as well because otherwise there would be a gap in the spectrum of h^0 by the definition of the relative particle number $N_L^0(E)$. This implies for an arbitrary $\epsilon > 0$ and L large enough

$$\begin{aligned} \left| \frac{N_L^0(E)}{L} - \frac{\#\{k : \lambda_k^L(0) \leq E\}}{L} \right| &\leq \left| \frac{\#\{k : (\frac{k\pi}{L})^2 \in (E - \epsilon, E + \epsilon)\}}{L} \right| \\ &\leq \frac{c}{\sqrt{E}} \epsilon, \end{aligned} \quad (4.7)$$

for some constant c . Since $\#\{k : \lambda_k^L(0) \leq E\}/L \rightarrow \rho_0(E)$, as $L \rightarrow \infty$, this yields the claim. \square

Given (4.1) and Lemma 4.1, Theorem 2.1 will follow from

Theorem 4.2. *Let $E > 0$. Then,*

$$\left| \det \left(\langle \varphi_j^L(0), \psi_k^L(0) \rangle \right)_{1 \leq j, k \leq N_L} \right|^2 = L^{-\zeta(E) + o(1)} \quad (4.8)$$

as $L \rightarrow \infty$, $N_L \in \mathbb{N}$ and $N_L/L \rightarrow \frac{\sqrt{E}}{\pi}$, where

$$\zeta(E) := \frac{1}{\pi^2} \delta_\alpha^2(\sqrt{E}) \quad (4.9)$$

and δ_α is given by Definition 2.2.

From now we shorten the notation and drop the 0 and L -index of the eigenfunctions and eigenvalues.

Apart from the product representation discussed in Section 3 the main ingredient to the proof of Theorem 4.2 is a elementary formula expressing the non-negative eigenvalues of the perturbed operator $h_{\alpha,L}^0$ in terms of the eigenvalues of the operator h_L^0 plus corrections depending on the scattering phase shift δ_α . First, note that the eigenvalues of h_L^0 can be computed explicitly, see [RS78], i.e. for $n \in \mathbb{N}$

$$\lambda_n = \left(\frac{n\pi}{L} \right)^2. \quad (4.10)$$

Lemma 4.3. *Let δ_α be given by Definition 2.2. Then,*

(i) *for $\alpha \geq 0$ and $n \in \mathbb{N}$ the n th eigenvalues of h_L^0 and $h_{\alpha,L}^0$ satisfy*

$$0 \leq \sqrt{\mu_n} = \sqrt{\lambda_n} - \frac{\delta_\alpha(\sqrt{\mu_n})}{L}, \quad (4.11)$$

(ii) *for $\alpha \leq 0$ and $n > 1$ the n th eigenvalues of h_L^0 and $h_{\alpha,L}^0$ satisfy*

$$0 \leq \sqrt{\mu_n} = \sqrt{\lambda_n} - \frac{\delta_\alpha(\sqrt{\mu_n})}{L}, \quad (4.12)$$

(iii) and δ exhibits the following expansion

$$\delta_\alpha(\sqrt{\mu_n}) = \delta_\alpha(\sqrt{\lambda_n}) - \frac{\delta'_\alpha(\sqrt{\lambda_n})\delta_\alpha(\sqrt{\lambda_n})}{L} + o\left(\frac{1}{L}\right), \quad (4.13)$$

which is valid for all $\mu_n \geq 0$, and the error term depends on α but is independent of n .

PROOF. Let $k > 0$. Consider the eigenvalue problem

$$-u_k'' = k^2 u_k, \quad -4\pi\alpha u_k(0+) + u_k'(0+) = 0. \quad (4.14)$$

Introducing Prüfer variables

$$u_k(x) = \rho_u(x) \sin(\theta_k(x)) \quad u_k'(x) = k\rho_u(x) \cos(\theta_k(x)), \quad (4.15)$$

we see that any non-zero solution of (4.14) is of the form

$$u_k(x) := a \sin\left(kx + \arctan\left(\frac{k}{4\pi\alpha}\right)\right), \quad (4.16)$$

for some $0 \neq a \in \mathbb{C}$. Since any eigenfunction u_k to an eigenvalue k^2 of $h_{\alpha,L}^0$ is a solution of (4.14) in $(0, L)$ and additionally satisfies $u_k(L-) = 0$, we obtain that

$$u_k(L) = a \sin\left(kL + \arctan\left(\frac{k}{4\pi\alpha}\right)\right) = 0. \quad (4.17)$$

On the other hand, all k^2 such that (4.17) is satisfied are eigenvalues of $h_{\alpha,L}^0$. Since the function $k \mapsto kL + \arctan\left(\frac{k}{4\pi\alpha}\right)$ is strictly increasing we obtain for any $n \in \mathbb{N}$ an unique eigenvalue $\mu_n \geq 0$ of $h_{\alpha,L}^0$ such that

$$\sqrt{\mu_n}L + \arctan\left(\frac{\sqrt{\mu_n}}{4\pi\alpha}\right) = n\pi, \quad (4.18)$$

where $\mu_1 < \mu_2 < \dots$. This proves (i). For the case $\alpha < 0$ note that $h_{\alpha,L}^0$ admits a single negative eigenvalue. Therefore, (4.18) is only valid starting from the second eigenvalue of $h_{\alpha,L}^0$. This implies for all $n \in \mathbb{N}$

$$\sqrt{\mu_{n+1}} = \sqrt{\lambda_n} - \frac{\arctan\left(\frac{\sqrt{\mu_{n+1}}}{4\pi\alpha}\right)}{L} = \sqrt{\lambda_{n+1}} - \frac{\pi - \arctan\left(\frac{\sqrt{\mu_{n+1}}}{4\pi|\alpha|}\right)}{L}. \quad (4.19)$$

(iii) follows directly from (i), (ii) and Definition (2.2) from the phase shift. \square

Corollary 4.4. *The eigenvalues of h_L^0 and $h_{\alpha,L}^0$ satisfy*

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \dots. \quad (4.20)$$

PROOF. Note that $|\delta_\alpha(k)| < \pi$ for all $k > 0$. Thus, (4.10) and (4.12) imply the corollary. \square

Next we apply the results from Section 3 to the determinant:

Lemma 4.5. *Let $N \in \mathbb{N}$. Then,*

$$\left| \det\left(\langle \varphi_j, \psi_k \rangle\right)_{1 \leq j, k \leq N} \right|^2 = \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{|\mu_k - \lambda_j| |\lambda_k - \mu_j|}{|\lambda_k - \lambda_j| |\mu_k - \mu_j|}. \quad (4.21)$$

PROOF. First note that $h_{\alpha,L}^0$ is bounded from below by (2.7). This and $h_L^0 \geq 0$ imply $-E \in \rho(h_L) \cap \rho(h_{\alpha,L}^0)$ for some $E > 0$. Moreover, (2.7) provides

$$\frac{1}{h_L^0 + E} - \frac{1}{h_{\alpha,L}^0 + E} = |\eta_L^{E,\alpha}\rangle\langle\eta_L^{E,\alpha}|, \quad (4.22)$$

for some $\eta_E^L \in L^2((0, L))$ and Corollary 4.4 gives

$$\frac{1}{\mu_1 + E} > \frac{1}{\lambda_1 + E} > \frac{1}{\mu_2 + E} > \frac{1}{\lambda_2 + E} > \dots, \quad (4.23)$$

the eigenvalues satisfy assumption (3.2). Furthermore, the operators $\frac{1}{h_L^0 + E}$ and $\frac{1}{h_{\alpha,L}^0 + E}$ are non-negative with trivial kernel and compact. Therefore, we are in position to apply Theorem 3.1 and obtain

$$\begin{aligned} \left| \det \left(\langle \varphi_j, \psi_k \rangle \right)_{1 \leq j, k \leq N} \right|^2 &= \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{\left| \frac{1}{\mu_k + E} - \frac{1}{\lambda_j + E} \right| \left| \frac{1}{\lambda_k + E} - \frac{1}{\mu_j + E} \right|}{\left| \frac{1}{\lambda_k + E} - \frac{1}{\lambda_j + E} \right| \left| \frac{1}{\mu_k + E} - \frac{1}{\mu_j + E} \right|} \\ &= \prod_{j=1}^N \prod_{k=N+1}^{\infty} \frac{|\mu_k - \lambda_j| |\lambda_k - \mu_j|}{|\lambda_k - \lambda_j| |\mu_k - \mu_j|}. \end{aligned} \quad (4.24)$$

□

PROOF OF THEOREM 4.2. We start with the product representation given in Lemma 4.5. Note that for $\alpha < 0$ there is an ambiguity since there exists precisely one negative eigenvalue μ_1 . Therefore, we treat the $j = 1$ term in the product separately. We define

$$A_L^N := \prod_{k=N+1}^{\infty} \frac{|\mu_k - \lambda_1| |\lambda_k - \mu_1|}{|\lambda_k - \lambda_1| |\mu_k - \mu_1|} = \prod_{k=N+1}^{\infty} \left| 1 + \frac{(\mu_k - \lambda_k)(\lambda_1 - \mu_1)}{(\lambda_k - \lambda_1)(\mu_k - \mu_1)} \right| \quad (4.25)$$

and estimate using Corollary 4.4

$$\begin{aligned} \sum_{k=N+1}^{\infty} \left| \frac{(\mu_k - \lambda_k)(\lambda_1 - \mu_1)}{(\lambda_k - \lambda_1)(\mu_k - \mu_1)} \right| &\leq |\lambda_1 - \mu_1| \sum_{k=N+1}^{\infty} \frac{\left(\left(\frac{k\pi}{L} \right)^2 - \left(\frac{(k-1)\pi}{L} \right)^2 \right)}{\left(\left(\frac{k\pi}{L} \right)^2 - \left(\frac{\pi}{L} \right)^2 \right) \left(\left(\frac{(k-1)\pi}{L} \right)^2 - \left(\frac{\pi}{L} \right)^2 \right)} \\ &\leq \frac{L^2}{\pi^2} |\lambda_1 - \mu_1| \sum_{k=N+1}^{\infty} \frac{(2k-1)}{(k^2-1)(k^2-2k)} \\ &\leq c \left(\frac{L}{N} \right)^2 |\lambda_1 - \mu_1|. \end{aligned} \quad (4.26)$$

Since h_L^α is uniformly bounded from below with respect to L , see Lemma 2.9,

$$\ln A_L^N = \ln \left(\prod_{k=N+1}^{\infty} \frac{|\mu_k - \lambda_1| |\lambda_k - \mu_1|}{|\lambda_k - \lambda_1| |\mu_k - \mu_1|} \right) = O(1) \quad (4.27)$$

as $N, L \rightarrow \infty$ and $\frac{N}{L} \rightarrow \rho(E) > 0$. Therefore, we are left with a product consisting of the non-negative eigenvalues and apply Lemma 4.5, use Lemma 4.3 (i) and $\sqrt{\lambda_n} =$

$\frac{n\pi}{L}$, $n \in \mathbb{N}$, to obtain

$$\begin{aligned} & \ln \left| \det \left(\langle \varphi_j, \psi_k \rangle \right)_{1 \leq j, k \leq N} \right|^2 = \ln A_L^N \\ & + \sum_{j=2}^N \sum_{k=N+1}^{\infty} \ln \left(\frac{ \left| (k\pi - \delta(\sqrt{\mu_k}))^2 - (j\pi)^2 \right| \left| ((k\pi))^2 - (j\pi - \delta(\sqrt{\mu_j}))^2 \right| }{ \left| (k\pi)^2 - (j\pi)^2 \right| \left| (k\pi - \delta(\sqrt{\mu_k}))^2 - (j\pi - \delta(\sqrt{\mu_j}))^2 \right| } \right). \end{aligned} \quad (4.28)$$

In the following the $O(1)$ and $o(1)$ terms refer to the asymptotics $L, N \rightarrow \infty$, $N/L \rightarrow \rho_0(E) > 0$. Equation (4.27) above, Lemma A.1 below and the abbreviation $g_k := -\frac{1}{\pi}\delta(\sqrt{\mu_k})$ for $k \in \mathbb{N}$ yield

$$(4.28) = - \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{(2jg_j + g_j^2)(2kg_k + g_k^2)}{((k + g_k)^2 - (j + g_j)^2)(k^2 - j^2)} + O(1). \quad (4.29)$$

Using Lemma A.2 and the abbreviation $\delta_k := -\frac{1}{\pi}\delta(\sqrt{\lambda_k})$ for $k \in \mathbb{N}$, we have

$$(4.29) = - \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2)}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} + O(1). \quad (4.30)$$

Lemma A.3 implies

$$(4.30) = - \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk\delta_j\delta_k}{(k^2 - j^2)^2} + O(1). \quad (4.31)$$

Lemma A.4 yields

$$(4.31) = -\frac{1}{\pi^2} \int_0^{\frac{N}{L}} dx \int_{\frac{N+1}{L}}^{\frac{2N}{L}} dy \frac{4xy\delta_\alpha(x\pi)\delta_\alpha(y\pi)}{(y^2 - x^2)^2} + O(1). \quad (4.32)$$

We define for $0 \leq x < y$

$$g(x, y) := \frac{4xy\delta_\alpha(\pi x)\delta_\alpha(\pi y)}{(y + x)^2} \quad (4.33)$$

The explicit representation of δ_α implies for all $\epsilon > 0$

$$\sup_{b > \epsilon} \sup_{(x, y) \in (0, b) \times (b, \infty)} \|(\nabla g)(x, y)\|_2 := c(\epsilon) < \infty. \quad (4.34)$$

Therefore, using the mean value theorem and the Cauchy-Schwarz inequality, we compute for a $0 < \epsilon < \sqrt{E}$ and N, L big enough

$$\begin{aligned} & \int_0^{\frac{N}{L}} dx \int_{\frac{N+1}{L}}^{\frac{2N}{L}} dy \left| \frac{4xy\delta_\alpha(x\pi)\delta_\alpha(y\pi)}{(y + x)^2} - \delta_\alpha^2(N/L) \right| \frac{1}{(y - x)^2} \\ & \leq c(\epsilon) \int_0^{\frac{N}{L}} dx \int_{\frac{N+1}{L}}^{\frac{2N}{L}} dy \| (N/L - x, y - N/L) \|_2 \frac{1}{(y - x)^2} \\ & \leq 2c(\epsilon) \int_0^{\frac{N}{L}} dx \int_{\frac{N+1}{L}}^{\frac{2N}{L}} dy \frac{1}{(y - x)} = O(1), \end{aligned} \quad (4.35)$$

where we used the inequality

$$\frac{|x - N/L| + |y - N/L|}{(y - x)^2} \leq 2 \frac{1}{(y - x)}, \quad (4.36)$$

which is valid for all $x < N/L < y$. Moreover, since $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi} > 0$, we compute

$$\int_0^{\frac{N}{L}} dx \int_{\frac{N}{L} + \frac{1}{L}}^{\frac{2N}{L}} dy \frac{1}{(y-x)^2} = \ln L + O(1). \quad (4.37)$$

Hence, combining equation (4.35) and (4.37), we end up with

$$(4.32) = -\ln L \frac{1}{\pi^2} \delta_\alpha^2(\pi N/L) + O(1) \quad (4.38)$$

$$= -\ln L \frac{1}{\pi^2} \delta_\alpha^2(\sqrt{E}) + o(\ln L), \quad (4.39)$$

where the last line follows from $\pi \frac{N}{L} \rightarrow \sqrt{E}$. This gives the assertion. \square

Appendix A. Proof of the auxiliary lemmata

In this section we prove the missing lemmata used in the proof of Theorem 4.2. We do not claim to give optimal or very elegant estimates. Throughout this section we drop the index α in the scattering phase shift and restrict ourselves to the case $\alpha < 0$. This implies the following estimate on the phase shift

$$\delta(x) - \delta(y) \geq 0, \quad (A.1)$$

for $x < y$, which we use in the sequel. The case $\alpha \geq 0$ is even simpler since in that case the Definition (2.2) of the phase shift implies the uniform bound

$$\|\delta\|_\infty \leq \frac{\pi}{2}, \quad (A.2)$$

which simplifies some of the following estimates. Moreover, we use the elementary asymptotics

$$\sum_{j=1}^N \sum_{k=N+1}^{\infty} \frac{1}{(k-j)^2} = O(\ln N), \quad (A.3)$$

$$\sum_{j=1}^N \sum_{k=N+1}^{\infty} \frac{1}{(k-j)^\beta} = O(1) \quad (A.4)$$

as $N \rightarrow \infty$, where $\beta > 2$.

Lemma A.1. *Set $g_k := -\frac{1}{\pi} \delta(\sqrt{\mu_k})$ for $k \in \mathbb{N}$. Then,*

$$\sum_{j=2}^N \sum_{k=N+1}^{\infty} \ln \left(\frac{((k+g_k)^2 - j^2)(k^2 - (j+g_j)^2)}{((k+g_k)^2 - (j+g_j)^2)(k^2 - j^2)} \right) \quad (A.5)$$

$$= - \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{(2jg_j + g_j^2)(2kg_k + g_k^2)}{((k+g_k)^2 - (j+g_j)^2)(k^2 - j^2)} + O(1) \quad (A.6)$$

as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$.

PROOF. We prove the assertion in two steps. First we consider the $j = N$ and $k = N + 1$ summand. Note that Lemma 4.1 above and $E > 0$ imply

$$\lim_{\substack{N, L \rightarrow \infty \\ N/L \rightarrow \sqrt{E}/\pi}} g_N = \lim_{\substack{N, L \rightarrow \infty \\ N/L \rightarrow \sqrt{E}/\pi}} g_{N+1} = -\frac{\delta(\sqrt{E})}{\pi} > -1. \quad (\text{A.7})$$

Thus, for $j = N$ and $k = N + 1$

$$\begin{aligned} & \lim_{\substack{N, L \rightarrow \infty \\ N/L \rightarrow \sqrt{E}/\pi}} \ln \left(\frac{((N+1+g_{N+1})^2 - N^2)((N+1)^2 - (N+g_N)^2)}{((N+1+g_{N+1})^2 - (N+g_N)^2)((N+1)^2 - N^2)} \right) \\ &= \lim_{\substack{N, L \rightarrow \infty \\ N/L \rightarrow \sqrt{E}/\pi}} \ln \left(\frac{(1+g_{N+1})(1-g_N)}{(1+g_{N+1}-g_N)} \frac{(2N+1+g_{N+1})(2N+1+g_N)}{(2N+1+g_{N+1}+g_N)(2N+1)} \right) \\ &= \ln \left(1 - \frac{\delta^2(\sqrt{E})}{\pi^2} \right). \end{aligned} \quad (\text{A.8})$$

Moreover, along the same line using (A.7)

$$\lim_{\substack{N, L \rightarrow \infty \\ N/L \rightarrow \sqrt{E}/\pi}} -\frac{(2Ng_N + g_N^2)(2(N+1)g_{N+1} + g_{N+1}^2)}{((N+1+g_{N+1})^2 - (N+g_N)^2)((N+1)^2 - N^2)} = -\frac{\delta^2(\sqrt{E})}{\pi^2}. \quad (\text{A.9})$$

Therefore, the $j = N$ and $k = N + 1$ term is of order 1.

For $j \leq N < N + 1 < k$ we want to apply the bound

$$|\ln(1+x) - x| \leq \frac{x^2}{2} \frac{1}{1-|x|} \quad (\text{A.10})$$

for $x \in \mathbb{R}$ with $|x| < 1$, to $x = x_{jk}$ where

$$x_{jk} := -\frac{(2jg_j + g_j^2)(2kg_k + g_k^2)}{((k+g_k)^2 - (j+g_j)^2)(k^2 - j^2)}. \quad (\text{A.11})$$

We estimate using $|g_n| \leq 1$ for all $n \in \mathbb{N}$ and $g_k - g_j \geq 0$

$$\begin{aligned} |x_{jk}| &\leq \left| \frac{(2j+g_j)(2k+g_k)}{(j+g_j+k+g_k)(k+j)} \right| \left| \frac{1}{(k-j+g_k-g_j)(k-j)} \right| \\ &\leq 2 \frac{1}{(k-j)^2}. \end{aligned} \quad (\text{A.12})$$

Since $j \leq N < N + 1 < k$, this implies in particular $|x_{jk}| \leq \frac{1}{2}$, and we continue using (A.10) and (A.12)

$$\begin{aligned} \sum_{j=1}^N \sum_{k=N+2}^{\infty} |\ln(1+x_{jk}) - x_{jk}| &\leq \sum_{j=1}^N \sum_{k=N+2}^{\infty} x_{jk}^2 \\ &\leq \sum_{j=2}^N \sum_{k=N+1}^{\infty} 4 \left(\frac{1}{k-j} \right)^4 = O(1), \end{aligned} \quad (\text{A.13})$$

as $N \rightarrow \infty$, where we used (A.4) in the last line. \square

Lemma A.2. Define $\delta_k := -\frac{1}{\pi}\delta(\sqrt{\lambda_k})$ for $k \in \mathbb{N}$. Then,

$$\sum_{j=2}^N \sum_{k=N+1}^{\infty} \left| \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2)}{((k + \delta_k)^2 - (j + \delta_j)^2)} - \frac{(2jg_j + g_j^2)(2kg_k + g_k^2)}{((k + g_k)^2 - (j + g_j)^2)} \right| \frac{1}{(k - j)^2} = o(1) \quad (\text{A.14})$$

as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$.

PROOF. First, using the expansion of Lemma 4.3, we obtain for all $n \in \mathbb{N}$, $n > 1$,

$$|g_n - \delta_n| \leq \frac{1}{\pi} |\delta(\sqrt{\mu_n}) - \delta(\sqrt{\lambda_n})| \leq \frac{\|\delta\|_{\infty} \|\delta'\|_{\infty}}{\pi L} := \frac{c}{L}, \quad (\text{A.15})$$

where the constant $c > 0$ depends only on α . We prove the assertion in two steps. In the first step we consider the numerator only in the second step we consider the denominator. Using (A.15) we estimate

$$\begin{aligned} & \sum_{j=2}^N \sum_{k=N+1}^{\infty} \left| \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2) - (2jg_j + g_j^2)(2kg_k + g_k^2)}{((k + g_k)^2 - (j + g_j)^2)(k^2 - j^2)} \right| \\ & \leq \frac{C}{L} \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{(j+1)(k+1)}{((k + g_k)^2 - (j + g_j)^2)(k^2 - j^2)} \\ & \leq \frac{C}{L} \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{(j+1)(k+1)}{(k+j-2)(k+j)(k-j)^2} = O\left(\frac{\ln N}{L}\right) \end{aligned} \quad (\text{A.16})$$

as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$, where we used $|g_j + g_k| \leq 2$, $g_k - g_j > 0$ for $j < k$ and (A.3). In order to estimate the denominator we use (A.15) to obtain some constant $c > 0$ independent of j, k such that

$$\left| ((k + g_k)^2 - (j + g_j)^2) - ((k + \delta_k)^2 - (j + \delta_j)^2) \right| \leq c \frac{k+j}{L}. \quad (\text{A.17})$$

Thus,

$$\begin{aligned} & \sum_{j=2}^N \sum_{k=N+1}^{\infty} (2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2) \left| \frac{1}{((k + g_k)^2 - (j + g_j)^2)(k^2 - j^2)} - \frac{1}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} \right| \\ & \leq \frac{4c}{L} \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{jk(k+j)}{(k^2 - j^2)^2 ((k + g_k)^2 - (j + g_j)^2) ((k + \delta_k)^2 - (j + \delta_j)^2)} \\ & \leq \frac{4c}{L} \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{jk}{(k-j)^4 (k+j-2)^2 (k+j)} = o(1) \end{aligned} \quad (\text{A.18})$$

as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$, where we used $|g_k + g_j| \leq 2$, $|\delta_k + \delta_j| \leq 2$, $g_k - g_j > 0$ and $\delta_k - \delta_j > 0$ for $j < k$. \square

Lemma A.3. *The estimate*

$$\left| \sum_{j=2}^N \sum_{k=N+1}^{\infty} \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2)}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} - \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk\delta_j\delta_k}{(k^2 - j^2)^2} \right| = O(1) \quad (\text{A.19})$$

holds as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$.

PROOF. First, we bound the tail, i.e. using $\delta_k - \delta_j > 0$ for $k > j$ and $|\delta_n| \leq 1$ for all $n \in \mathbb{N}$ we estimate

$$\begin{aligned} & \sum_{j=2}^N \sum_{k=2N+1}^{\infty} \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2)}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} \leq \sum_{j=2}^N \sum_{k=2N+1}^{\infty} \frac{1}{(k - j)^2} \\ & \leq \sum_{k=2N+1}^{\infty} \frac{N}{(k - N)^2} = O(1), \end{aligned} \quad (\text{A.20})$$

as $N \rightarrow \infty$. We insert $\pm \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk\delta_j\delta_k}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)}$ in (A.19). Thus, in the next step $\delta_k - \delta_j > 0$ yields

$$\begin{aligned} & \sum_{j=2}^N \sum_{k=N+1}^{2N} \left| \frac{(2j\delta_j + \delta_j^2)(2k\delta_k + \delta_k^2) - 4jk\delta_j\delta_k}{((k + \delta_k)^2 - (j + \delta_j)^2)(k^2 - j^2)} \right| \\ & \leq \sum_{j=2}^N \sum_{k=N+1}^{2N} \left| \frac{2(k + j) + 1}{(k - j)^2(k + j)(k + j - 2)} \right| \\ & \leq 3 \sum_{j=2}^N \sum_{k=N+1}^{2N} \left| \frac{1}{(k - j)^2(k + j - 2)} \right| = O\left(\frac{\ln N}{N}\right), \end{aligned} \quad (\text{A.21})$$

as $N \rightarrow \infty$, where we used (A.3) in the last line. In the third step, again $|\delta_n| \leq 1$ for $n \in \mathbb{N}$ yields

$$\begin{aligned} & \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk}{(k^2 - j^2)} \left| \frac{1}{((k + \delta_k)^2 - (j + \delta_j)^2)} - \frac{1}{(k^2 - j^2)} \right| \\ & \leq \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{9jk(k + j)}{(k^2 - j^2)^2(k + j - 2)(k - j)} \\ & \leq 9 \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{1}{(k - j)^3} = O(1), \end{aligned} \quad (\text{A.22})$$

as $N \rightarrow \infty$, where we used (A.4). \square

Lemma A.4. *The asymptotics*

$$\left| \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk\delta_j\delta_k}{(k^2 - j^2)^2} - \frac{1}{\pi^2} \int_0^{\frac{N}{L}} dx \int_{\frac{N+1}{L}}^{\frac{2N}{L}} dy \frac{4xy\delta(x\pi)\delta(y\pi)}{(y^2 - x^2)^2} \right| = O(1) \quad (\text{A.23})$$

holds as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$.

PROOF. We recall that $\delta_k := -\frac{1}{\pi}\delta(\sqrt{\lambda_k})$ and we rewrite

$$\sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4jk\delta_j\delta_k}{(k^2-j^2)^2} = \frac{1}{L^2\pi^2} \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{4\frac{j}{L}\frac{k}{L}\delta(\frac{j\pi}{L})\delta(\frac{k\pi}{L})}{((\frac{k}{L})^2 - (\frac{j}{L})^2)^2}. \quad (\text{A.24})$$

Thus, we estimate

$$\begin{aligned} & \left| \frac{1}{L^2} \sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{\frac{j}{L}\frac{k}{L}\delta(\frac{j\pi}{L})\delta(\frac{k\pi}{L})}{((\frac{k}{L})^2 - (\frac{j}{L})^2)^2} - \int_{\frac{1}{L}}^{\frac{N}{L}} dx \int_{\frac{N+1}{L}}^{\frac{2N+1}{L}} dy \frac{xy\delta(x\pi)\delta(y\pi)}{(y^2-x^2)^2} \right| \\ & \leq \sum_{j=2}^N \sum_{k=N+1}^{2N} \int_{\frac{j-1}{L}}^{\frac{j}{L}} dx \int_{\frac{k}{L}}^{\frac{k+1}{L}} dy \left| f\left(\frac{j}{L}, \frac{k}{L}\right) - f(x, y) \right|, \end{aligned} \quad (\text{A.25})$$

where

$$f(x, y) := \frac{xy\delta(x\pi)\delta(y\pi)}{(y^2-x^2)^2}. \quad (\text{A.26})$$

Using the mean-value theorem and the Cauchy-Schwarz inequality we obtain

$$\begin{aligned} (\text{A.25}) & \leq \sum_{j=2}^N \sum_{k=N+1}^{2N} \sup_{(x,y) \in (\frac{j-1}{L}, \frac{j}{L}) \times (\frac{k}{L}, \frac{k+1}{L})} |(\nabla f)(x, y)|_2 \\ & \quad \times \int_{\frac{j-1}{L}}^{\frac{j}{L}} dx \int_{\frac{k}{L}}^{\frac{k+1}{L}} dy \left| \left(\frac{j}{L} - x, \frac{k}{L} - y \right) \right|_2 \\ & \leq \frac{1}{L^3} \sum_{j=2}^N \sum_{k=N+1}^{2N} \sup_{(x,y) \in (\frac{j-1}{L}, \frac{j}{L}) \times (\frac{k}{L}, \frac{k+1}{L})} |(\nabla f)(x, y)|_2, \end{aligned} \quad (\text{A.27})$$

where $|\cdot|_2$ denotes the Euclidean norm. We compute

$$(\nabla f)(x, y) = \frac{1}{(y^2-x^2)^3} \quad (\text{A.28})$$

$$\begin{aligned} & \times \left((y^2-x^2)(y\delta(x\pi)\delta(y\pi) + xy\delta'(x\pi)\delta(y\pi)\pi) + 4x^2y\delta(x\pi)\delta(y\pi) \right) \\ & \times \left((y^2-x^2)(x\delta(x\pi)\delta(y\pi) + xy\delta(x\pi)\delta'(y\pi)\pi) - 4xy^2\delta(x\pi)\delta(y\pi) \right) \\ & =: \frac{1}{(y^2-x^2)^3} g(x, y). \end{aligned} \quad (\text{A.29})$$

We estimate for $(x, y) \in (\frac{j-1}{L}, \frac{j}{L}) \times (\frac{k}{L}, \frac{k+1}{L})$, $j \leq N < k$,

$$\left(\frac{1}{y^2-x^2} \right)^3 \leq \frac{L^6}{(k+j-1)^3(k-j)^3} \leq \frac{L^6}{N^3} \frac{1}{(k-j)^3} \quad (\text{A.30})$$

and, using $\delta, \delta' \in L^\infty((0, \infty))$,

$$\sup_{(x,y) \in (\frac{j-1}{L}, \frac{j}{L}) \times (\frac{k}{L}, \frac{k+1}{L})} |g(x, y)|_2 \leq \sup_{(x,y) \in (0, \frac{2N+1}{L}) \times (0, \frac{2N+1}{L})} |g(x, y)|_2 = O(1) \quad (\text{A.31})$$

as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$. Thus, (A.30) and (A.31) imply

$$(\text{A.27}) \leq O\left(\sum_{j=2}^N \sum_{k=N+1}^{2N} \frac{1}{(y-x)^3} \right) = O(1) \quad (\text{A.32})$$

as $N, L \rightarrow \infty$, $\frac{N}{L} \rightarrow \frac{\sqrt{E}}{\pi}$. □

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MATHEMATISCHES INSTITUT, LUDWIG-MAXIMILIANS-UNIVERSITÄT MÜNCHEN, THERESIENSTRASSE 39, 80333 MÜNCHEN, GERMANY

E-mail address: gebert@math.lmu.de